Problem 11

Using (4.6), give a proof of the preliminary test. *Hint:* $S_n - S_{n-1} = a_n$.

Solution

Consider a generic infinite series.

$$\sum_{i=0}^{\infty} a_i$$

The partial sum is

$$S_n = \sum_{i=0}^{n} a_i = \sum_{i=0}^{n-1} a_i + a_n = S_{n-1} + a_n.$$

Bring S_{n-1} to the left side.

$$S_n - S_{n-1} = a_n$$

Take the limit of both sides as $n \to \infty$.

$$\lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} a_n$$

Assume that the infinite series converges and use equation (4.6).

$$S - S = \lim_{n\to\infty} a_n$$

Evaluate the left side.

$$0 = \lim_{n\to\infty} a_n \tag{1}$$

If

$$\lim_{n\to\infty} a_n \neq 0,$$

then equation (1) is false, meaning the assumption that the infinite series converges is false.

$$\lim_{n\to\infty} S_n \neq S$$

Therefore, if

$$\lim_{n\to\infty} a_n \neq 0,$$

the infinite series diverges.